Universal features of information spreading efficiency on *d*-dimensional lattices

E. Agliari,[1] R. Burioni,[1,2] D. Cassi,[1,2] and F. M. Neri[1]
[1]*Dipartimento di Fisica, Università degli Studi di Parma, viale Usberti 7/A, 43100 Parma, Italy*
[2]*INFN, Gruppo Collegato di Parma, viale Usberti 7/A, 43100 Parma, Italy*
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A model for information spreading in a population of N mobile agents is extended to d-dimensional regular lattices. This model, already studied on two-dimensional lattices, also takes into account the degeneration of information as it passes from one agent to the other. Here, we find that the structure of the underlying lattice strongly affects the time τ at which the whole population has been reached by information. By comparing numerical simulations with mean-field calculations, we show that dimension $d=2$ is marginal for this problem and mean-field calculations become exact for $d>2$. Nevertheless, the striking nonmonotonic behavior exhibited by the final degree of information with respect to N and the lattice size L appears to be geometry independent.

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I. INTRODUCTION

The problem of information spreading among a population has been intensively studied in the last years and several aspects have been focused upon [1–4].

The population is generally represented by means of a graph such that an interaction (link) between two or more agents (nodes) means that there is a flow of information among them. Recently, the dynamics of agents making up the population has also been taken into account [1,2]. Not only does the mobility of agents provide a realistic feature, but it also affects the network of acquaintances.

In an earlier paper [1] we introduced a model where agents are represented by N random walkers which diffuse on a square $L\times L$ lattice and possibly interact if they are close together. This model also takes into account the degradation of information when passing from one agent to another: a decay constant z quantifies such alteration. As a consequence, the information spreading is a history-dependent process which is governed by the rules underlying the diffusion of N random walkers on the given space.

Indeed, diffusion phenomena are dramatically affected by the topology of the underlying space [5–9]. For example, on infinite lattices, the asymptotic probability for a random walker to return to its starting point equals 1 in one and two dimensions, while for $d\geqslant 3$ there is a non-null probability to never return to the starting point. The former case is said to be *recurrent*, while the latter is called *transient* [5–7]. Moreover, in many processes concerning *interacting* random walkers, dimension 2 plays the role of an upper critical dimension: it separates a higher-dimensional regime where the mean-field results are exact from a lower-dimensional regime where fluctuations become important. This is the case, for example, for two-species diffusion-limited reactions [8], or for the trapping of a random walker by diffusing traps [9]. In such processes, additional logarithmic corrections for the power laws in dimension $d=2$ typically appear. Hence, in general, when dealing with diffusion one should also wonder how the laws describing the problem are affected by the geometry.

The model introduced in Ref. [1] for $d=2$ is now extended to d-dimensional hypercubic lattices. Numerical simulations are carried for dimensions from $d=1$ up to $d=5$. Analytical investigations are led which especially focus on the one-dimensional case and on a mean-field approach which provides good estimates for high-dimensional ($d\geqslant 3$) lattices. Most of the results presented here do also hold for general Euclidean (i.e., translationally invariant) lattices, since the large-scale topology of these systems (and quantities depending on it, such as the time τ defined below in the low-density limit) depends solely on their dimension, and not on small-scale details.

The main important quantities we are concerned with are the population-awareness time τ, which represents the average time necessary for the piece of information to reach the whole population, and the final degree of information per agent $\mathcal{I}_{ag}(z)$.

The time τ depends on the system parameters N and L. Our numerical results show that in the low-density regime, and in every dimension d, this dependence can be factorized as $\tau(N,L)=f(N)g(L)$, where f and g depend on d. Moreover, in the low-density regime, we find the asymptotic behaviors of both $f(N)$ and $g(L)$ to agree with mean-field calculations for $d\geqslant 3$, while for $d=2$ deviations from the mean-field behavior appear and for $d=1$ the results are radically different. We therefore argue that dimension $d=2$ is marginal for the phenomenon under examination, and the mean-field calculation of τ is exact for $d>2$.

The most important result contained in Ref. [1] concerns a nonasymptotic phenomenon: the nonmonotonic dependence of the final degree of information per agent $\mathcal{I}_{ag}$ on N and L, with the emergence of extremal points. A process of optimization of the final information is therefore intrinsically nontrivial. We show here that the existence of extremal points is not a consequence of the special choice $d=2$, but it arises in all dimensions $d\geqslant 1$. It therefore appears as a *universal* and geometry-independent phenomenon, occurring at the crossover between high- and low-density regimes.

The paper is organized as follows. Section II is devoted to the description of the model. Section III contains analytical results; it is divided into high-density calculations (Sec. III A); low-density calculations for $d=1$ (Sec. III B); low-density calculations for $d>1$ (Sec. III C). Section IV shows results obtained by means of numerical simulations. We first consider the population awareness time τ (Sec. IV A), then the behavior of the final degree of information and the quantities that affects it (Secs. IV B and IV C). Finally, Sec. V includes our conclusions and perspectives.

II. THE MODEL

The model analyzed in this work represents an extension of the one introduced in an earlier paper [1]. In this section we briefly recall how it works.

We consider a population of N random walkers (agents) moving on a d-dimensional hypercubic lattice sized L and endowed with periodic boundary condition. Agents are initially ($t=0$) distributed randomly throughout the whole volume L^d. We define the density of agents as $\rho=N/L^d$; the "low-density" regime is for $\rho \ll 1$ and the "high-density" regime for $\rho \gg 1$. At each following instant each agent jumps randomly to one of the $2d$ nearest-neighbor sites. Notice that the same site can be occupied by more agents, i.e, there are no excluded-volume effects.

At $t=0$ we assume that only one agent (called "Information Source") carries information, while the remaining $N-1$ agents are unaware. Two agents can then interact if their distance on the underlying lattice is $\leqslant 1$ and if one of them is informed and the other unaware. By "interaction" we mean an information passing from the informed agent, say j, to the unaware one k with a fixed decay constant z $(0 \leqslant z \leqslant 1)$: if j carries information I_j, then k becomes informed with information $I_k = zI_i$. Hence, the information carried by agent j is represented by the quantity I_j, $0 \leqslant I_j \leqslant 1$, and, in particular, when $I_j > 0$ $(I_j=0)$ the agent is "aware" ("unaware").

Once an agent has become informed, it will never change nor lose its information. As a consequence, there exists a final time t_{fin} at which the total information of the system can no longer evolve: at this time the information has reached every agent and the simulation stops. Such time is a stochastic quantity with average value τ called the population-awareness time (PAT) and standard deviation denoted as σ_τ. Part of this work is devoted to studying the properties of τ as a function of L and N.

The total number of informed agents at a given time t is again a stochastic variable; we call $n(t)$ its average over all the realizations of the system $[n(0)=1;\, n(\infty)=N]$. As a result of our model, the information carried by an agent is always a power z^l of the decay constant, l being the number of passages from the Information Source to the agent. It is convenient to divide informed agents into levels, so that an agent belongs to level l when the information it receives has undergone l passages from the Information Source and equals z^l. We call $n(l,t)$ the number of agents belonging to the lth level at time t, averaged over all different realizations $[n(t) = \Sigma_{l=0}^{t} n(l,t)]$. The average total information at time t is therefore the generating function of $n(t)$,

$$\mathcal{I}(z,t) = \sum_{l=0}^{t} n(l,t) z^l. \qquad (1)$$

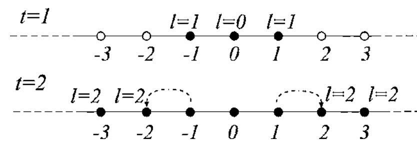

FIG. 1. High-density approximation for $d=1$.

In particular, we are interested in the final degree of information $\mathcal{I}(z)$, that is the total information achieved once the whole population has been informed,

$$\mathcal{I}(z) = \mathcal{I}(z,\infty) = \sum_{l=0}^{N} n(l,\infty) z^l. \qquad (2)$$

We also denote its average value per agent as $\mathcal{I}_{ag}(z) = \mathcal{I}(z)/N$. The quantity $n(l,\infty)$ as a function of l is called the final distribution of the population on levels.

III. ANALYTICAL RESULTS

Although the model cannot be exactly solved in the general case, it is possible to provide approximate solutions in some limit cases. There are two time scales involved in the process: one for the diffusion of the random walkers and one for the information passing. When they are very different, approximate analytical approaches become feasible and give results in good agreement with the numerical simulations. When the two time scales are comparable, only a numerical approach is possible (Sec. IV).

In this section we give analytical results for the PAT and the final distribution on levels in two limit cases. Section III A treats the high-density $(\rho \gg 1)$ limit, with particular regard to the case $d=1$. Section III B considers the asymptotic low-density $(\rho \ll 1)$ limit for $d=1$. A general mean-field theory of this limit for all dimensions is given in Sec. III C.

A. High-density regime

When $\rho \gg 1$, we can assume that the set of informed agents covers a connected volume of the lattice, and that this volume expands with a constant velocity (depending on the density ρ and dimension d). We clarify this statement by considering $d=1$.

Let us consider a chain of finite length L, with $N \to \infty$ agents on it, and label the sites with the numbers from 1 to L (Fig. 1). The number of agents on a given site is ρ; for $N \to \infty$ $(\rho \to \infty)$, the probability that a given site is empty is 0. Also, for $\rho \to \infty$, we assume that with probability 1 at least one of the ρ agents of a given site will jump to the left and one to the right. Let the source be in 0 at $t=1$: all the agents in 0, 1, and -1 will get informed. At $t=2$ some of the newly informed agents jump on ± 2; hence, the agents on ± 2 and ± 3 become informed. The motion of the information front decouples from the random motion of the agents; it expands with a *deterministic* law, with constant velocity of 4 sites per time unit. The time required to cover the whole chain is then

$$\tau \simeq L/4. \qquad (3)$$

The border of the informed zone contains all the newly informed agents, so each time step adds a new level and

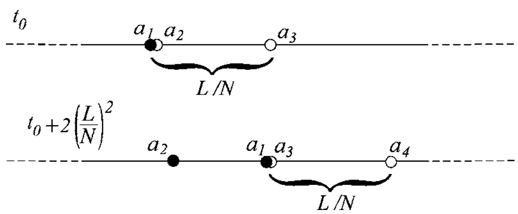

FIG. 2. Low-density approximation for $d=1$.

$n(l,\infty)=4\rho$ for $l\geq 1$. This is the origin, for high densities, of the plateau observed in the one-dimensional distributions (Fig. 7).

For $d>1$ it can be shown (see Ref. [1] for the case $d=2$) that the volume of the informed zone is a d-dimensional polyhedron. The time it takes the border of the informed volume to reach the border of the lattice is again $L/4$, but now $L/4$ more instants are required to cover the rest of the lattice. Hence, in this case

$$\tau \simeq L/2. \tag{4}$$

The new agents added at each step cover a $(d-1)$-dimensional surface, hence $n(l,\infty)\simeq l^{d-1}$ for $l\leq L/4$, and $n(l-L/4,\infty)\simeq n(L/4,\infty)-(l-L/4)^{d-1}$ for l up to $L/2$.

B. Low-density regime in $d=1$

Let us consider a chain of length L and a population of N agents randomly distributed on it, with $N \ll L$ (Fig. 2). The average distance among two agents is $\rho^{-1}=\frac{L}{N}$. Due to the low-density hypothesis, we can neglect interactions involving more than two agents. In this approach we divide the problem of diffusion among N agents into a sum of easier (three-bodies) problems.

Let us consider the instant t_0 when the rightmost agent a_1 informs an unaware agent a_2 (Fig. 2). Let us call a_3 the next unaware agent on the right: the average distance from a_3 to a_1 and a_2 is $L/N=\rho^{-1}$ (if ρ is small enough, we can consider a_1 and a_2 to be on the same site). A calculation regarding an epidemic model in one dimension (1D) similar to ours [11] found that for low densities the velocity of the front propagation approaches $\rho/2$. Hence, the average time it takes one of the two aware agents to meet the unaware one is $(L/N)/(\rho/2)=2(L/N)^2$. If we now suppose that a_3 has been first reached by a_2, it will take again a time $2(L/N)^2$ for one of them to reach the next unaware agent a_4 on the right, and so on. There are about $N/2$ processes of this kind on the right-hand side and $N/2$ on the left-hand side, which provides

$$\tau \sim \frac{L^2}{N}. \tag{5}$$

Now, if a_1 belongs to level l (a_2 to level $l+1$), a_3 will belong to levels $l+1$ or $l+2$ with probabilities 1/2; a_4 will belong to levels $l+1$, $l+2$ or $l+3$ with probabilities 1/4, 1/2, and 1/4, respectively, and so on. It is easy to show by induction, starting from a source on level 0, that at the ith information passing the new agent on the left-hand side is on level l with probability $2^{-i}\binom{i}{l}$ $(0\leq l\leq i)$; the same for the new agent on the right-hand side. Hence, the average final number of agents on level l is

$$n(l,\infty) \sim \sum_{i=0}^{N/2} 2^{-i}\binom{i}{l}. \tag{6}$$

There is no easy closed form for this sum, but it can be plotted for any value of N: the curve displays a plateau of height 2, before decaying to 0.

If we call $\Delta\mathcal{I}(t)$ the increment of the total information at time t, we can write

$$\Delta\mathcal{I}(i+1) = \Delta\mathcal{I}(i)\frac{z+1}{2}, \tag{7}$$

and therefore

$$\mathcal{I}(z) = \frac{z(z+1)}{2(1-z)}\left[1-\left(\frac{z+1}{2}\right)^{N/2}\right]. \tag{8}$$

C. Low-density regime in $d>1$

In the case of low density ($\rho \ll 1$) the time an informed agent walks before meeting an unaware agent becomes very large. We adopt a mean-field approximation by assuming that the agents between each event have the time to redistribute randomly on the lattice. In this approximation, the probability that two given agents are in contact at a given time is $p_d=\langle\tau_d\rangle^{-1}$, where $\langle\tau_d\rangle$ is the average time for two random walkers to meet on a d-dimensional cubic lattice.

The process is an absorbing Markov chain with N states; the system is in state k when it has k informed agents. The chain starts from state 1 and evolves to the absorbing state (state N). The transition matrix $\mathbf{P}$ can be written: the transition probability from a state k to a state m as a function of N and p_d is

$$P_{km} = \binom{N-k}{m-k}[1-(1-p_d)^k]^{m-k}[(1-p_d)^k]^{N-m}$$

for any N and p_d. This is an upper triangular matrix, since the binomial coefficient $\binom{N-k}{m-k}$ is 0 for $m<k$. We then make a low-density approximation: we expand matrix $\mathbf{P}$ to first order in p_d to obtain

$$P_{km} = \begin{cases} 1-m(N-m)p_d & \text{for } m=k, \\ m(N-m)p_d & \text{for } m=k+1, \\ 0 & \text{elsewhere}. \end{cases}$$

This means that the system in the state m has a probability $1-m(N-m)p_d$ to stay in m and a probability $m(N-m)p_d$ to jump to state $m+1$. We now take matrix $\mathbf{Q}$, the submatrix obtained from $\mathbf{P}$ subtracting the last row and column (those pertaining to the absorbing state), and compute the fundamental matrix $\mathbf{F}=(1-\mathbf{Q})^{-1}$; a direct calculation shows that $\mathbf{F}$ is an upper triangular matrix given by

$$F_{km} = \begin{cases} \dfrac{1}{m(N-m)p_d} & \text{for } k \geq m, \\ 0 & \text{for } k < m. \end{cases}$$

The mean time τ required to reach the absorbing state N starting from state 1 is given by the sum of the first row of $\mathbf{F}$,

$$\tau = \frac{1}{p_d} \sum_{m=1}^{N-1} \frac{1}{m(N-m)}, \tag{9}$$

and for $N \to \infty$,

$$\tau \sim \frac{2}{Np_d}[\gamma + \ln(N)] = 2\langle \tau_d \rangle \frac{\gamma + \ln(N)}{N}, \tag{10}$$

where $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant.

A classical result [5,10] states that for d-dimensional cubic lattices the asymptotic dependence of $\langle \tau_d \rangle$ on the lattice size L is

$$\langle \tau_d \rangle \sim \begin{cases} u_1 L^2, & d=1, \\ u_2 L^2 \ln(L), & d=2, \\ u_d L^d, & d>2, \end{cases}$$

where the u_d are dimension-dependent constants (for example, $u_2 = 0.758\ldots$). As we will show in the following, the asymptotic dependence of τ on L agrees with mean-field results for *every* d, while the breakdown of the mean-field theory shows up in the dependence on N for $d \leq 2$.

It is possible to include the distribution on levels in the Markov chain analysis, but the calculations become very lengthy and we give only the final result. It is

$$n(l, \infty) = \frac{1}{(N-1)!} |s(N, l+1)|, \tag{11}$$

and is exact for every N. Here, $|\cdots|$ denotes the absolute value, and $s(m,k)$ is the Stirling number of the first kind [12]. The $s(m,k)$ are integers that appear in many combinatorial problems; one of the possible asymptotic expansions for Stirling numbers is [13]

$$\frac{1}{(m-1)!} |s(m,k)| = \gamma \frac{\ln (m)^{k-1}}{(k-1)!} + O[\ln (m)^{k-2}],$$

so that

$$n(l, \infty) \sim \frac{\ln (N)^l}{l!}, \tag{12}$$

which is the form we will use to fit the low-density distributions. From this distribution it also follows that

$$\mathcal{I} \sim N^z. \tag{13}$$

IV. NUMERICAL RESULTS

A. Population-awareness time

In this section we focus on numerical results concerning the population-awareness time τ. We recall that τ has been defined as the average time it takes the piece of information to reach the whole population. Due to the analytical results discussed in the preceding section, we expect the functional form displayed by $\tau(N,L)$ to be strongly affected by the topology of the lattice underlying the propagation.

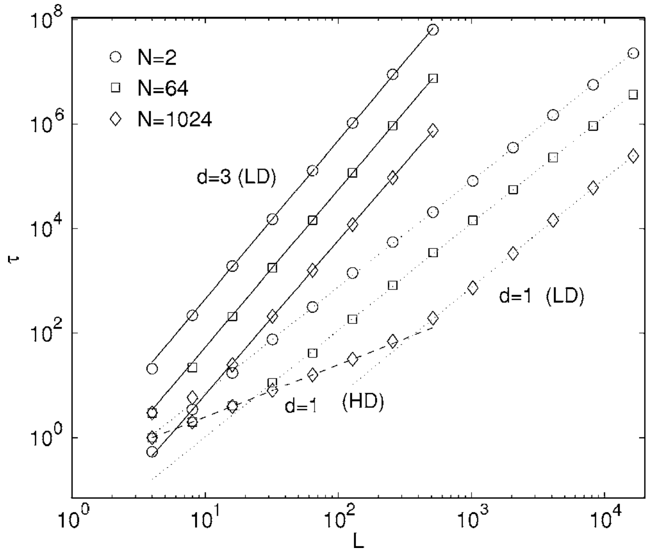

FIG. 3. Log-log scale plot of the population-awareness time τ versus the lattice size L. Results obtained for a chain (dashed line) and for a cube (solid line) are depicted. Different lattice-size values are shown with different symbols, as explained by the legend. For large densities ($\rho \gg 1$, HD) and low densities ($\rho \ll 1$, LD), straight lines represent the best fit according to Eq. (14) and Eqs. (15) and (16), respectively. Error on data points is less than 2%. The standard deviation is not appreciable on this scale.

Figures 3 and 4 show the dependence of τ on L with N fixed, and on N with L fixed, respectively. In Fig. 3, where both results for $d=1$ and $d=3$ are displayed, two different

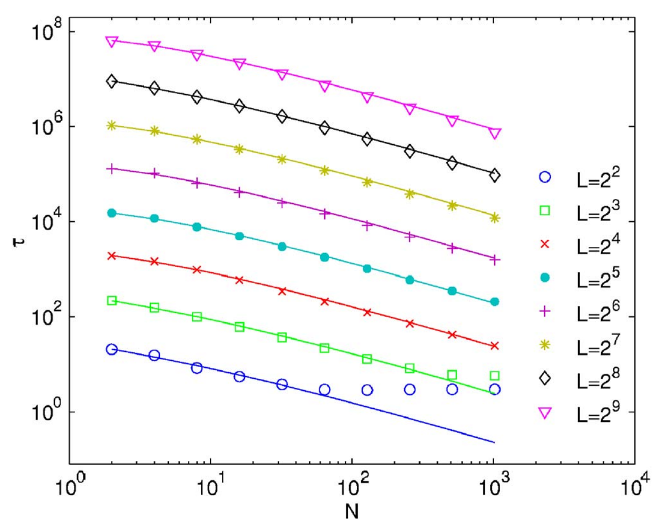

FIG. 4. (Color online) Dependence of the population-awareness time τ on the number of agents N for the cubic lattice $d=3$; different values of the lattice size L are shown with different symbols and colors. When the density of the system is low, data points lie on the curve given by Eq. (16) which represents the best fit. Error on data points is less than 2%.

regimes, of low and high density, are clearly distinguishable for dimension $d=1$; for $d=3$ the range of the high-density regime is too small, and only the low-density one can be seen.

The high-density behavior is independent of N; indeed, for $\rho \gg 1$ we find

$$\tau = \frac{L}{4}, \quad d = 1,$$

$$\tau = \frac{L}{2}, \quad d > 1, \tag{14}$$

in agreement with Eqs. (3) and (4).

In the low-density regime ($\rho \ll 1$) and for $d \neq 2$, τ follows the behavior calculated in Secs. III B and III C. For $d=1$ the results as a function of L and N are fitted by

$$\tau \sim C_1 N^{\alpha} L^{\beta}, \tag{15}$$

with $C_1=0.96(5)$; $\alpha=-0.98(5)$; $\beta=1.98(2)$, in agreement with Eq. (5).

For $d=3,4$ we found the best fit for τ to be given by the function

$$\tau \sim C_d \frac{\ln(L) + A}{N} L^{\beta}, \tag{16}$$

where $\beta=d$ within a 1% error and $A=0.59(3)$, in agreement with the value $\gamma=0.577\ldots$ found in Eq. (10). The values of the constants are $C_3=0.77(1)$; $C_4=0.39(1)$ (different in general from the u_d of the mean-field approximation).

The low-density limit in the case $d=2$ deserves a separate discussion. It is still possible to express τ as a product of two distinct functions,

$$\tau \sim f_2(N) L^2 \ln(L) \quad \text{for } d = 2. \tag{17}$$

The dependence on L is in agreement with the improved mean-field calculation (see Sec. III C). This best fit is better than that in Ref. [1], where we hypothesized a noninteger power law ($L^{2.2}$).

The analytical form of $f_2(N)$ cannot be unequivocally determined by the simulations. In the fitting range the function $\frac{A+\ln(N)}{N}$ agrees with the numerical results better than the power-law $N^{-0.66}$ previously given [1]. However, in this case the value of the fitting parameter is $A=-0.18(2)$, hence is definitely different from the mean-field value γ. Since there are no analytical calculations to support this functional form with this particular value of the fitting constant for $d=2$, we cannot rule out higher-order logarithmic corrections.

To summarize, in the low-density regime, the function $\tau(N,L)$ factorizes into two parts, depending, respectively, on L and N,

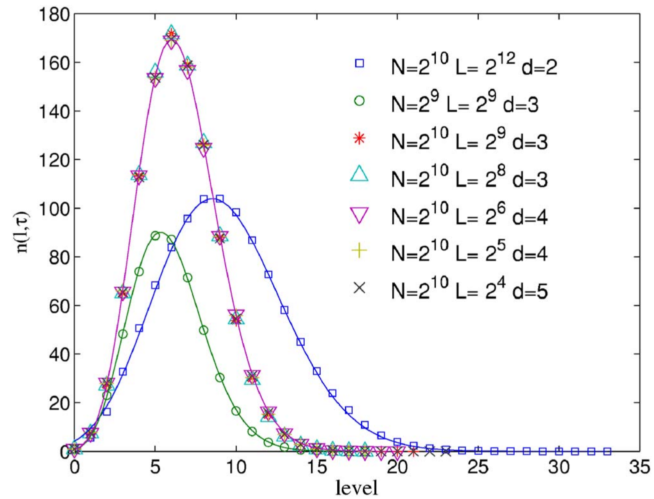

FIG. 5. (Color online) Final population distribution on levels $n(l,\infty)$ for low-density systems. Data points agree with the fitting line drawn according to Eq. (19). For $d=2$ the fitting parameters depend smoothly on both L and N, and the curve is distinct from those of higher dimension and same N. For $d \geq 3$, systems of different dimension d and size L display distributions that overlap within the error. Only the dependence on N is left, as is shown for $d=3$, $N=512$.

$$\tau \sim \begin{cases} C_1 \dfrac{L^2}{N}, & d = 1, \\ f_2(N) L^2 \ln(L), & d = 2, \\ C_d \dfrac{\gamma + \ln(N)}{N} L^d, & d \geq 3, \end{cases} \tag{18}$$

the C_d being dimension-depending constants. The most satisfying fitting function we have found for $f_2(N)$ is $\frac{A+\ln(N)}{N}$, $A \simeq -0.18$.

The standard deviation σ_τ displays a similar dependence on N and L for low densities: $\sigma_\tau \sim N^{-1} L^2$ for $d=1$, and so on. For high densities σ_τ becomes vanishingly small, which is explained by the fact that the propagation of information becomes a deterministic process.

B. Final distribution on levels: Universality of the extremal distribution

In Sec. II we introduced the function $n(l,\infty)$, which represents the final distribution of agents on levels and is strongly connected with the final degree of information $\mathcal{I}(z)$. The asymmetrical-bell shape displayed by the distributions for hypercubic lattices with dimension $d>2$ (Fig. 5) and the way they evolve while varying the system parameters N and L are analogous to the two-dimensional case [1].

In the limit case $\rho \gg 1$ (Fig. 6) and in every dimension, the final distribution on levels follows the law $n(l,\infty) \sim l^{d-1}$, in agreement with the calculation in Sec. III A.

For ρ small enough (i.e., for $L > \tilde{L}$ and $N < \tilde{N}$, see below) the population distribution on levels for $d \geq 2$ is well fitted by the following function:

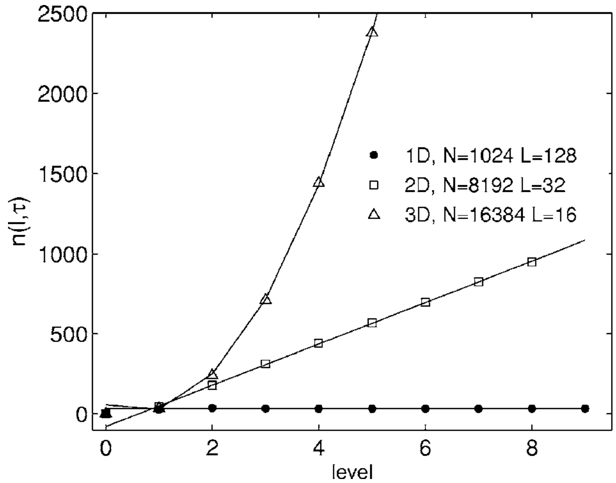

FIG. 6. Final population distribution on levels for high densities and $d=1,2,3$. The dependence on l is a power law, $n(l,\infty)\sim l^{d-1}$.

$$\frac{n(l,\infty)}{N}=A\frac{(\ln N)^l}{\Gamma(Bl+C)},\qquad(19)$$

where $\Gamma(x)$ is the Euler gamma function (Fig. 5). The previous equation is a generalization of Eq. (12) found in the mean-field approximation for the low-density regime.

The fitting parameters A,B,C at low densities for $d=2$ are smoothly dependent on N and L, while those for $d>2$ are all close to 1 and independent of the lattice size L, keeping only the dependence on N. Moreover, the data points for $d\geqslant 3$ with the same values of N *all collapse on the same curve*. This holds for all the regular lattices with $d\geqslant 3$ we have considered: the distribution curves at low densities are independent of d, and agree with the mean-field form [Eq. (12)].

The description given so far concerns $d\geqslant 2$ lattices; in Fig. 7 we show for $N=512$ and varying L the one-dimensional case, which exhibits quite different distributions. Such distributions are still very sharp for very high values of the density ρ, but soon develop a plateau by increasing L; the plateau persists up to low densities. The existence of a plateau was justified both in a high-density and in a low-density approximation (Sec. III).

The main result of Ref. [1] concerned the existence of an *extremal* curve for the distribution of agents on levels. We have found that this feature does not depend on the dimension d of the lattice. We show in Fig. 7 how the extremal distribution emerges in $d=1$, as a function of L, for a particular value $L=\widetilde{L}$ (here, $\widetilde{L}\simeq 1024$), and keeping N fixed, notwithstanding the fact that its shape is dramatically different with respect to the higher-dimensional ones. While $L<\widetilde{L}$, the distribution displays a plateau whose height (width) is a monotonically decreasing (increasing) function of the chain length L; the distribution curve shifts to the right. Conversely, for $L>\widetilde{L}$ a shift-back phenomenon analogous to that discussed in Ref. [1]: now, by rising L, the height gets larger while the width gets smaller. As can be seen from Fig. 3, $\widetilde{L}$ corresponds to the crossover between high- and low-density regimes. The same happens by varying N and keeping L fixed; there is an extremal distribution for a particular value $\widetilde{N}$, depending on L, and corresponding to the crossover between the two regimes.

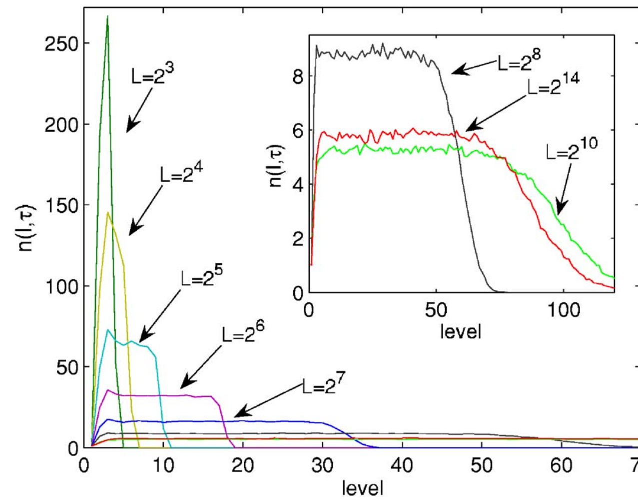

FIG. 7. (Color online) Population distribution on levels at $t=\tau$ for one-dimensional systems with $N=512$ and L ranging from 2^3 to 2^{14}, as shown by the legend (the lines are guides to the eye). The behavior of the distribution is nonmonotonic with respect to L: by increasing L from small values, the curves first shift to the right and flatten; the rightmost, extremal curve corresponds to $L=1024$. Then, by further increasing L, the curves shift back to the left and sharpen. The inset shows in detail the shift back with the curves pertaining to $L=2^8, L=2^{10}, L=2^{14}$.

This shift-back phenomenon, and the existence of an extremal distribution, occur in all the dimensions we have investigated (up to $d=5$). It therefore constitutes a universal feature, independent of lattice dimension, and, as we will see, it provides striking effects on the final degree of information.

C. Degree of information

In this section we deal with the final degree of information $\mathcal{I}(z)=\mathcal{I}(z,\infty)$ (2) and its dependence on the decay constant z and system parameters N, L. We remind [Eq. (1)] that $\mathcal{I}(z)$ is the generating function of the final populations $n(l,\infty)$, hence its value depends on the final distribution of the population on levels analyzed in the preceding paragraphs.

Let us first consider the dependence on the decay constant z. Again, results highlight strong differences between the one-dimensional and higher-dimensional cases ($d\geqslant 2$). In the latter case and for the low-density regime (approximately $\rho<2^{-8}$), we find

$$\mathcal{I}(z)=N^z,\qquad(20)$$

within the error ($<4\%$).

On the other hand, when $d=1$, the final degree of information shows an exponential growth which can be represented by the following equation:

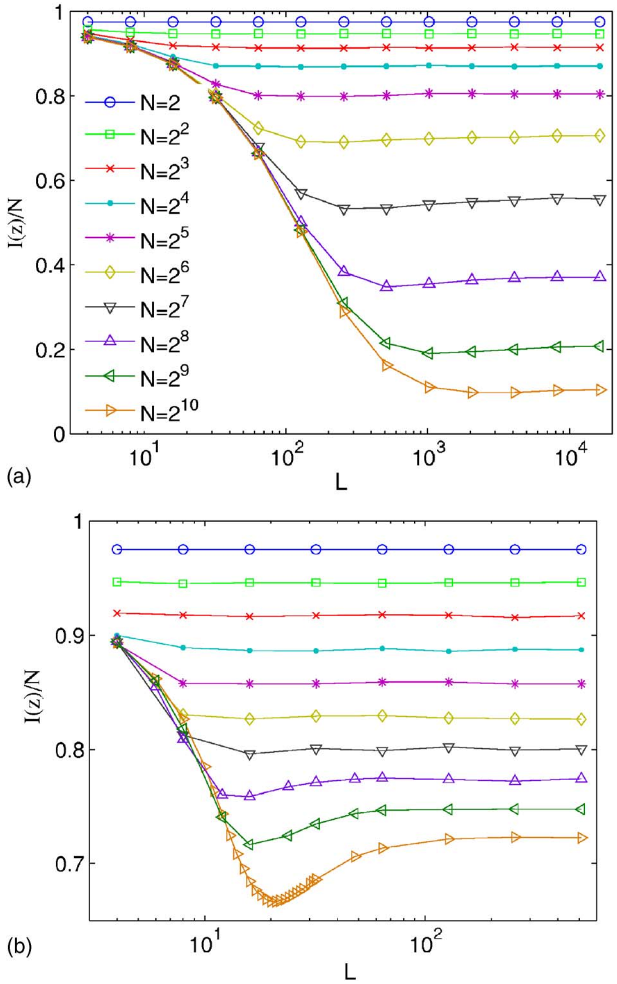

FIG. 8. (Color online) Semilog scale plot of final degree of information per agent $\mathcal{I}_{ag}(z)=\mathcal{I}(z)/N$ vs lattice size L, for $d=1$ (top) and $d=3$ (bottom). The decay constant is fixed at $z=0.95$. Several values of N are shown with different symbols and colors (lines are guides to the eye) and the legend is the same for both figures. Notice that minimum depth is greater for the latter case. Error on data points is $<4\%$.

$$\mathcal{I}(z) = A\frac{z(1+z)}{1-z}(1-e^{-BN(1-z)}), \tag{21}$$

where A and B smoothly depend on N and L. Equations (20) and (21) are in very good agreement with the expressions found in the mean-field approximation [Eqs. (13) and (8), respectively].

Once z is fixed, $\mathcal{I}_{ag}(z)$ depends nonmonotonically on N and L: let us follow it for N fixed and varying L in Fig. 8 in the two cases $d=1$ and $d=3$. For L small, due to the narrow distribution discussed in the preceding section, the value of the information is high. When $L=\tilde{L}$, the population distribution on levels reaches its extremal form and the information displays a minimum. As L increases, the information starts to rise again (as can be seen, the effect gets more marked by increasing the dimension). Hence, given a population number N, there is an optimal lattice size $\tilde{L}$ for which the final information is minimum. The same happens having fixed L and letting N vary: there is a minimum for $N=\tilde{N}$, depending on L. As underlined in Ref. [1], the existence of a local minimum of the final information implies that choosing an optimization strategy for the spreading of information on the lattice is not trivial. There is no *a priori* right direction in parameter space where to move in order to improve $\mathcal{I}(z)$; rather, the direction depends on the starting point.

V. CONCLUSIONS AND PERSPECTIVES

In this work the model of information spreading previously introduced has been extended to different geometries; indeed, we considered the chain and d-dimensional hypercubic lattices. The occurrence of a nonmonotonic behavior for the final degree of information is not due to a special geometry underlying the process, but its origin lies in the crossover between the two different regimes of high and low density. Therefore, the existence of minima in the final degree of information is universal and, remarkably, even the possibility to derive optimization strategies does not depend on the particular structure the process is embedded in.

On the other hand, the asymptotic laws for τ are interestingly related to the geometry underlying the random-walk diffusion. In particular, $d=2$ is a marginal dimension separating two well-behaved cases, which suggests an investigation on in-between dimensions [14].

The robustness of the existence of extremal point for $\mathcal{I}$ is an important point since the possibility of extracting optimal strategies is not a feature restricted to some special structures.

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2, can be mapped onto the problem of absorption at the origin with a proper rescaling of time.

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